MATHS WORKSHOPS

Probability, Sigma Notation and Combinatorics

Business School

THE UNIVERSITY OF SYDNEY

Welcome to the Business School Maths Workshops

- Aim: to familiarise you with the basic mathematics requirements for studying units in the Business School.
  - Basic algebra
  - Graphing and interpreting graphs
  - Inequalities
  - Simultaneous equations
  - Basic calculus
  - Factorial notation
  - Summation notation
  - Basic probability concepts and calculations
  - Simple and compound interest
- Throughout there are links that you can click on to find out more about a particular concept: ▸ More.
- Most ▸ More link to Wikipedia – never reference Wikipedia in your assignments – always find a more official source – but it is a good initial reference for mathematics and statistics.

Outline

Outline

Events

Definition (Sample Space)

The sample space, often denoted Ω, of an experiment or random trial is the set of all possible outcomes. ▸ More

Definition (Event)

An event, sometimes denoted ω, is a set of outcomes (a subset of the sample space) to which a probability is assigned. ▸ More

Example (Tossing a dice)

- The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$ because either a 1 or a 2 or a 3 or a 4 or a 5 or a 6 must be on the surface.
- If we are interested in rolling an even number, the event of interest is $\omega = \{2, 4, 6\}$.

More events

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Definition (Mutually exclusive)

Two (or more) events are mutually exclusive if they cannot occur at the same time.

▸ More

Example (Tossing a dice)

The events rolling a 2 and rolling a 3 are mutually exclusive because you cannot roll a 2 and a 3 at the same time.

Definition (Collectively exhaustive)

A set of events is collectively exhaustive if it encompasses all possible outcomes.

▸ More

Example (Tossing a dice)

The events 1, 2, 3, 4, 5 and 6 are collectively exhaustive because one of these must occur in each roll of the dice.

Probability

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Definition (Probability)

Probability is a way of expressing knowledge or belief about the likelihood of an event occurring. ▸ More

- Mathematically, the probability that some event, let's call it E, occurs is expressed as:

$$P(E) \quad \text{or} \quad Pr(E).$$

- The probability of an event occurring must be between 0 and 1:

$$0 \leq P(E) \leq 1$$

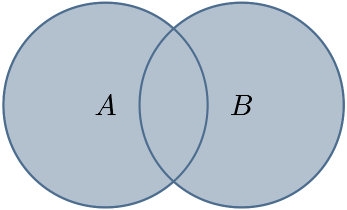
- If an event cannot happen it has probability, $P(E) = 0$.
- If an event is certain to happen, its probability is, $P(E) = 1$.

Union

Definition (Union)

If either event A or event B or both events occur at the same time, this is called the union of the events A and B. It is denoted as $A \cup B$.

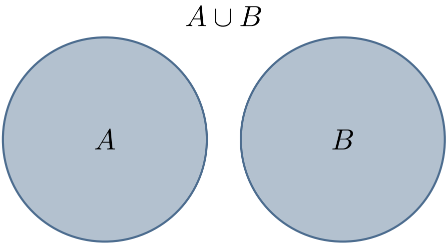
- $A \cup B$ is sometimes read as "A or B" but remember that $A \cup B$ really means "A or B or both A and B".
- Venn diagram representation:



More

Union of Mutually Exclusive Events

- Recall if two events are mutually exclusive then if one occurs, the other cannot occur.
- We can represent this in a Venn diagram where there's no overlap between the two events:



- Mathematically, if you have two (or more) events that are mutually exclusive then:

$$P(A \cup B) = P(A) + P(B)$$

Intersection

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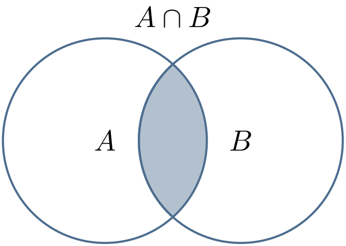
Definition (Intersection)

If both event A and event B occur at the same time, this is called the intersection of events A and B. It is denoted as

$$A \cap B.$$

- $A \cap B$ is sometimes read as "A and B."
- Venn diagram representation:



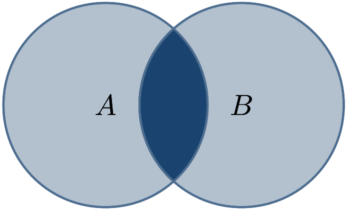
▸ More

Union of Non-Mutually Exclusive Events

- If events are not mutually exclusive then it is possible for them to both occur at the same time.
- Mathematically, if you have two (or more) events that are not mutually exclusive then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- The darker shaded area is $P(A \cap B)$.



- $P(A) + P(B)$ counts the overlapping section $P(A \cap B)$ twice!

Independence

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Definition (Independent)

Two events A and B are independent if the occurrence of event A makes it neither more nor less probable that event B occurs.

- Mathematically, independence occurs if and only if

$$P(A \cap B) = P(A) \times P(B)$$

More

Example (Throwing a dice)

The event of getting a 4 the first time a dice is rolled and the event of getting a 4 the second time are independent.

- The probability of rolling a 4 in the first roll and a 4 again in the second roll:

$$P(\text{first roll 4} \cap \text{second roll 4}) = P(4) \times P(4) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}.$$

Rolling a dice revisited

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Example (Your turn. . .)

In one roll of the dice:

- What is the probability of getting a 1?
- What is the probability of getting a 1 and a 3?
- What is the probability of getting an odd number?
- What is the probability of rolling a number at least as big as 5?

Rolling a dice revisited

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Example (Your turn. . .)

In one roll of the dice:

- What is the probability of getting a 1?
    - $P(1) = \frac{1}{6}$
- What is the probability of getting a 1 and a 3?
- What is the probability of getting an odd number?
- What is the probability of rolling a number at least as big as 5?

Rolling a dice revisited

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Example (Your turn...)

In one roll of the dice:

- What is the probability of getting a 1?
  - $P(1) = \dfrac{1}{6}$
- What is the probability of getting a 1 and a 3?
  - $P(1 \cap 3) = 0$ because they are mutually exclusive events.
- What is the probability of getting an odd number?
- What is the probability of rolling a number at least as big as 5?

Rolling a dice revisited

Example (Your turn...)

In one roll of the dice:

- What is the probability of getting a 1?
  - $P(1) = \frac{1}{6}$
- What is the probability of getting a 1 and a 3?
  - $P(1 \cap 3) = 0$ because they are mutually exclusive events.
- What is the probability of getting an odd number?
  - Because rolling 1, 3 and 5 are mutually exclusive events, $P(1 \cup 3 \cup 5) = P(1) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.
- What is the probability of rolling a number at least as big as 5?

Rolling a dice revisited

Example (Your turn. . .)

In one roll of the dice:

- What is the probability of getting a 1?
  - $P(1) = \dfrac{1}{6}$
- What is the probability of getting a 1 and a 3?
  - $P(1 \cap 3) = 0$ because they are mutually exclusive events.
- What is the probability of getting an odd number?
  - Because rolling 1, 3 and 5 are mutually exclusive events, $P(1 \cup 3 \cup 5) = P(1) + P(3) + P(5) = \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{6} = \dfrac{1}{2}$.
- What is the probability of rolling a number at least as big as 5?
  - $P(5 \cup 6) = P(5) + P(6) = \dfrac{1}{6} + \dfrac{1}{6} = \dfrac{2}{6} = \dfrac{1}{3}$.

Outline

Review of Probability

Sigma Notation

Permutations and Combinations

Conclusion

General notation for writing observations

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Definition (Observations)

For a general sample of size n we write the observations as $x_1, x_2, \ldots, x_n$. In other words, the ith observation is denoted x_i for $i = 1, 2, \ldots, n$.

Example (Observe the heights of 5 individuals)

Name	i	x_i
Jack	1	$x_1 = 175$cm
Jill	2	$x_2 = 163$cm
Xiao	3	$x_3 = 182$cm
Jim	4	$x_4 = 171$cm
Jane	5	$x_5 = 159$cm

Sigma Notation

Definition (Sigma Notation)

We write the sum of n observations as:

More

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n$$

- The summation operator, $\sum$, is the greek letter, capital sigma, hence the name "Sigma notation."
- The operator, $\sum_{i=1}^{n}$, is read as "the sum from $i = 1$ to n."
- You can use it to sum any number, not just observations:

$$\sum_{i=1}^{3} 1 = 1 + 1 + 1 = 3 \quad \text{or} \quad \sum_{i=1}^{4} a = a + a + a + a = 4a$$

Sigma Notation

Example (Observe the heights of 5 individuals)

Name	i	x_i
Jack	1	$x_1 = 175$cm
Jill	2	$x_2 = 163$cm
Xiao	2	$x_3 = 182$cm
Jim	4	$x_4 = 171$cm
Jane	5	$x_5 = 159$cm

The sum of these observations is:

$$\begin{aligned} \sum_{i=1}^{5} x_i &= x_1 + x_2 + x_3 + x_4 + 5_5 \\ &= 175 + 163 + 182 + 171 + 159 \\ &= 850 \end{aligned}$$

Your turn with Sigma Notation...

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Example (Suppose $\sum_{i=1}^{4} x_i = 12$)

- $\displaystyle\sum_{i=1}^{4} 2 =$
- $\displaystyle\sum_{i=1}^{4} 5x_i =$
- $\displaystyle\sum_{i=1}^{4} (5x_i + 2) =$
- $\displaystyle\frac{1}{4}\sum_{i=1}^{4} x_i =$

Your turn with Sigma Notation...

Example (Suppose $\sum_{i=1}^{4} x_i = 12$)

- $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 2 \times 4 = 8$
- $\displaystyle\sum_{i=1}^{4} 5x_i =$
- $\displaystyle\sum_{i=1}^{4} (5x_i + 2) =$
- $\displaystyle\frac{1}{4}\sum_{i=1}^{4} x_i =$

Your turn with Sigma Notation...

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Example (Suppose $\sum_{i=1}^{4} x_i = 12$)

- $$\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 2 \times 4 = 8$$
- $$\sum_{i=1}^{4} 5x_i = 5x_1 + 5x_2 + 5x_3 + 5x_4 = 5(x_1 + x_2 + x_3 + x_4)$$
  $$= 5 \times \sum_{i=1}^{4} x_i = 5 \times 12 = 60$$
- $$\sum_{i=1}^{4} (5x_i + 2) =$$
- $$\frac{1}{4} \sum_{i=1}^{4} x_i =$$

Your turn with Sigma Notation...

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Example (Suppose $\sum_{i=1}^{4} x_i = 12$)

- $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 2 \times 4 = 8$
- $\displaystyle\sum_{i=1}^{4} 5x_i = 5x_1 + 5x_2 + 5x_3 + 5x_4 = 5(x_1 + x_2 + x_3 + x_4)$

  $$= 5 \times \sum_{i=1}^{4} x_i = 5 \times 12 = 60$$
- $\displaystyle\sum_{i=1}^{4} (5x_i + 2) = \sum_{i=1}^{4} 5x_i + \sum_{i=1}^{4} 2 = 60 + 4 \times 2 = 68$
- $\displaystyle\frac{1}{4}\sum_{i=1}^{4} x_i =$

Your turn with Sigma Notation...

Example (Suppose $\sum_{i=1}^{4} x_i = 12$)

- $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 2 \times 4 = 8$
- $$\sum_{i=1}^{4} 5x_i = 5x_1 + 5x_2 + 5x_3 + 5x_4 = 5(x_1 + x_2 + x_3 + x_4)$$
  $$= 5 \times \sum_{i=1}^{4} x_i = 5 \times 12 = 60$$
- $\displaystyle\sum_{i=1}^{4} (5x_i + 2) = \sum_{i=1}^{4} 5x_i + \sum_{i=1}^{4} 2 = 60 + 4 \times 2 = 68$
- $\displaystyle\frac{1}{4}\sum_{i=1}^{4} x_i = \frac{1}{4} \times 12 = 3$

Outline

Permutations

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Definition (Permutation)

A permutation of a set of objects is an arrangement of the objects in a certain order.

More

Example (Pizza with pepperoni, onions and mushrooms)

Under the definition of a permutation, the following pizzas are all different:

- Pepperoni, onion, mushroom
- Onion, mushroom, pepperoni
- Mushroom, pepperoni, onion
- Onion, pepperoni, mushroom
- Mushroom, onion, pepperoni
- Pepperoni, mushroom, onion

Permutations without replacement

Example (How many different permutations are there of a pizza with pepperoni, onions and mushrooms)

To find the number of different arrangements:

1. Select a first choice from 3 possible choices.
2. Take a second choice; there are 2 choices remaining.
3. Finally, there is 1 choice for the last selection.

Thus, there are $3 \times 2 \times 1 = 6$ different ordered arrangements of the toppings. All of these were found on the previous slide.

Definition (Factorial)

The factorial of a positive integer, n, denoted by $n!$, is the product of all positive integers less than or equal to n:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

▸ More

Permutations without replacement

- What if you have a set of objects and only want to arrange part of them? I.e. a permutation of n objects r at a time.

Theorem

The number of permutations of a set of n objects taken r at a time is given by the following formula: ${}^nP_r = \dfrac{n!}{(n-r)!}$.

▸ More

Example (How many ways to arrange different 3 toppings on a pizza if there are 6 available?)

- You can select the first topping in 6 ways, the second in 5, and the third in 4. This can be written as $6 \times 5 \times 4$.
- Using the formula with $n = 6$ and $r = 3$ we get:

$${}^6P_3 = \frac{6!}{(6-3)!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 6 \times 5 \times 4 = 120.$$

Permutations without replacement

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Example (Your turn. . .)

If a university has lockers with 50 numbers on each combination lock, how many possible permutations using three numbers are there.

▸ More

- Recognise that
  - n, the number of objects, is
  - r, the number of objects taken at one time, is .
- Use those numbers in the permutation formula:

$$^nP_r = \qquad .$$

Permutations without replacement

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Example (Your turn. . .)

If a university has lockers with 50 numbers on each combination lock, how many possible permutations using three numbers are there.

More

- Recognise that
  - n, the number of objects, is 50
  - r, the number of objects taken at one time, is 3.
- Use those numbers in the permutation formula:

$$^{n}P_{r} = {}^{50}P_{3} = \frac{50!}{(50-3)!} = 50 \times 49 \times 48 = 117,600.$$

Permutations with replacement

Things are greatly simplified when you can repeat the objects.

Theorem

The number of arrangements of n objects taken r at a time, with repetition, is given by n raised to the power of r: n^r.

▸ More

Example

How many license plates can you make with only 4 letters on them, given that you can repeat the letters?

- You can take the first letter from 26 options
- You can also take the second letter from 26 options
- Same for the third and fourth letters.

Therefore, there are $26 \times 26 \times 26 \times 26 = 26^4 = 456,976$ available license plates using 4 letters if you can repeat letters.

Permutations with replacement

Example (Your turn. . .)

How many 4 digit license plates can you make using the numbers from 0 to 9 while allowing repetitions.

- Realise there are $n =$ objects taken $r =$ at a time.
- Plug that information into the formula:

$$n^r = \qquad .$$

Permutations with replacement

Example (Your turn...)

How many 4 digit license plates can you make using the numbers from 0 to 9 while allowing repetitions.

- Realise there are $n = 10$ objects taken $r = 4$ at a time.
- Plug that information into the formula:

$$n^r = 10^4 = 10 \times 10 \times 10 \times 10 = 10,000.$$

Combinations

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Definition (Combination)

Unordered arrangements of objects are called combinations.

Example

Under the definition of combinations, a pizza with the left half pineapple and the right half pepperoni is the same thing as a pizza with the left half pepperoni and the right half pineapple.

Theorem

The number of combinations of a set of n objects taken r at a time is given by: $^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

- There's a button for this on most calculators.

▸ More

Intuition behind the combination formula

Example (How many different types of pizzas are there if each pizza has 3 toppings out of a possible 6?)

- You can select the first topping in 6 ways, the second in 5, and the third in 4. This can be written as $6 \times 5 \times 4 = 120$.
- Formula: $n = 6$ and $r = 3$ we get: ${}^6P_3 = \frac{6!}{(6-3)!} = 120$.
- BUT this calculation is a permutation: it treats the order as important. We need to divide the number of permutations by the number of different ways of arranging the selections.
- There are $r! = 3 \times 2 \times 1 = 6$ ways of arranging 3 ingredients. Eg. there's 6 different ways to place Mushroom, onion and pepperoni on a pizza.
- So the formula is: $\frac{1}{r!} \times \frac{n!}{(n-r)!} = \frac{1}{6} \times 120 = 20$.

Combinations

Example (How many ways can you choose 4 people at random from a group of 10 people?)

Since you're going to have the same group of $r = 4$ people no matter what order you choose the people in, you set up the problem as a combination.

$$
\begin{aligned}
{}^{10}C_4 = \binom{10}{4} &= \frac{10!}{4!(10-4)!} \\
&= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)} \\
&= 210
\end{aligned}
$$

Thus, there are 210 different groups of $r = 4$ people you can choose from a larger group of $n = 10$.

Combinations

Example (Your turn. . .)

1. How many committees of 4 students can be chosen from a class of 30 students?
   - Order is unimportant here – dealing with a combination!
   - Total number of students, $n =$.
   - Number chosen, $r =$.
   - ${}^nC_r =$
2. If the Group of Eight University football teams all play each other exactly once, how many games are played?
   - Order is unimportant here – dealing with a combination!
   - Total number of universities, $n =$.
   - Number of teams playing in any given game, $r =$.
   - ${}^nC_r =$

Combinations

Example (Your turn...)

1. How many committees of 4 students can be chosen from a class of 30 students?
    - Order is unimportant here – dealing with a combination!
    - Total number of students, $n = 30$.
    - Number chosen, $r = 4$.
    - ${}^nC_r = {}^{30}C_4 = \binom{30}{4} = \frac{30!}{4!(30-4)!} = 27,405.$
2. If the Group of Eight University football teams all play each other exactly once, how many games are played?
    - Order is unimportant here – dealing with a combination!
    - Total number of universities, $n =$.
    - Number of teams playing in any given game, $r =$.
    - ${}^nC_r =$

Combinations

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Example (Your turn. . .)

1. How many committees of 4 students can be chosen from a class of 30 students?
   - Order is unimportant here – dealing with a combination!
   - Total number of students, $n = 30$.
   - Number chosen, $r = 4$.
   - $^nC_r = {}^{30}C_4 = \binom{30}{4} = \frac{30!}{4!(30-4)!} = 27,405.$
2. If the Group of Eight University football teams all play each other exactly once, how many games are played?
   - Order is unimportant here – dealing with a combination!
   - Total number of universities, $n = 8$.
   - Number of teams playing in any given game, $r = 2$.
   - $^nC_r = {}^{8}C_2 = \binom{8}{2} = \frac{8!}{2!(8-2)!} = 28.$

Permutations and Combinations Summary

- If the order doesn't matter, it is a combination.
- If the order does matter it is a permutation.

Permutations

- Repetition allowed: n^r
- No repetition: $\dfrac{n!}{(n-r)!}$

Combinations

- No repetition: $\dfrac{n!}{r!(n-r)!}$

More

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Applications in Business

- In Business Statistics probability concepts and summation notation are used extensively ▸ More
- In Insurance, probability concepts and the theory of permutations and combinations are used to determine the the premium you need to pay ▸ More
- In Finance the risk of an investment strategy is quantified using probability arguments ▸ More
- In Management often there will be a number of options and the one you pick may be based on the likelihood of success: determined using probability theory ▸ More
- In Business Information Systems risk management is often undertaken using probability arguments. ▸ More

Outline

Reading Greek Letters

Name	Symbol	Name	Symbol
alpha	α	nu	ν
beta	β	xi	ξ, Ξ
gamma	γ, Γ	omicron	o
delta	δ, Δ	pi	π, Π
epsilon	ϵ, ε	rho	ρ
zeta	ζ	sigma	σ, Σ
eta	η	tau	τ
theta	θ, Θ	upsilon	υ
iota	ι	phi	ϕ, Φ
kappa	κ	chi	χ
lambda	λ, Λ	psi	ψ, Ψ
mu	μ	omega	ω, Ω

Summary

- Sample spaces and events
- Probability statements: $P(E)$
- Intersections, unions and independence
- Permutations and combinations
- Denoting observations using x_i
- Sigma notation
- Sequences, series and limits
- Arithmetic and geometric progressions
- Sums of arithmetic and geometric progressions
- Superannuation

Coming up. . .

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Week 3: Algebra, Linear Equations and Series

- Parameters, variables and solving equations
- Understanding, solving and graphing linear equations
- Identifying and working with sequences and series

Week 4: Functions

- Understanding, solving and graphing Quadratic Functions
- Understanding Logarithmic and Exponential Functions

Coming up. . .

Week 5:Simultaneous Equations and Inequalities

- Algebraic and graphical solutions to simultaneous equations
- Understanding and solving inequalities

Week 6: Differentiation

- Theory and rules of Differentiation
- Differentiating various functions and application of Differentiation

Additional Resources

- Test your knowledge at the University of Sydney Business School MathQuiz:
  http://quiz.econ.usyd.edu.au/mathquiz
- Additional resources on the Maths in Business website
  sydney.edu.au/business/learning/students/maths
- The University of Sydney Mathematics Learning Centre has a number of additional resources:
  - Basic concepts in probability notes ▸ More
  - Sigma notation notes ▸ More
  - Permutations and combinations notes ▸ More
  - Further workshops by the Maths Learning Centre ▸ More
- There's also tonnes of theory, worked questions and additional practice questions online. All you need to do is Google the topic you need more practice with! ▸ More

Acknowledgements

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- Questions, comments, feedback? Let us know at `business.maths@sydney.edu.au`