MATHS WORKSHOPS

Differentiation

Business School

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Outline

The theory of differentiation

How differentiation is done in practice

Application: Finding Maxima and Minima

Conclusion

Outline

The theory of differentiation

How differentiation is done in practice

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Conclusion

Differentiation

- A useful way to explore the properties of a function is to find the derivative.

Definition (Derivative)

The derivative is a measure of how a function changes as its input changes.

More

- The derivative of a function at a chosen input value describes the best linear approximation of the function near that input value.
- For single variable functions, $f(x)$, the derivative at a point equals the slope of the tangent line to the graph of the function at that point.
- The process of finding a derivative is called differentiation.

Tangent

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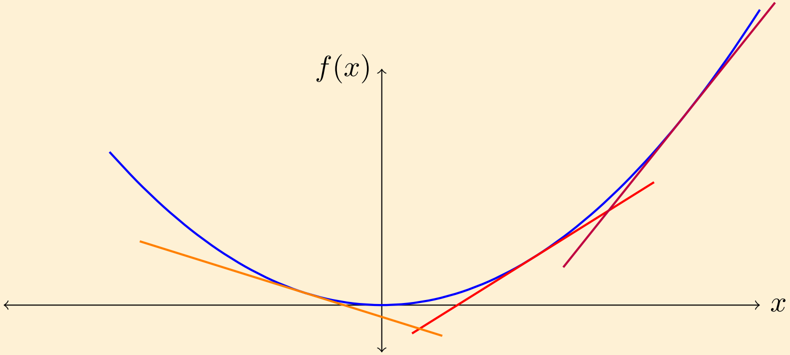
Definition (Tangent)

The tangent to a curve at a given point is the straight line that "just touches" the curve at that point. More

Example

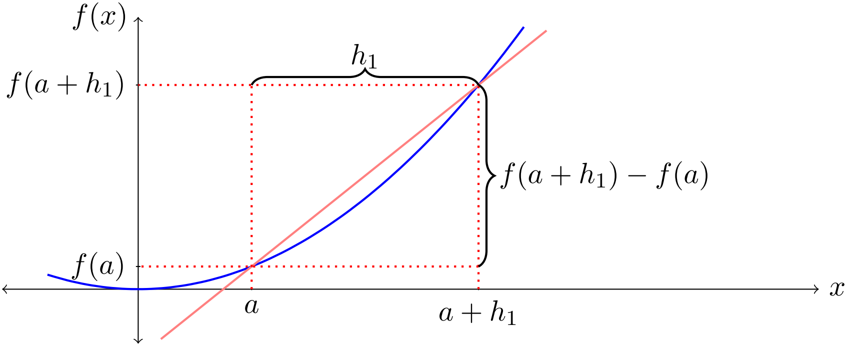


Formal Statement of Differentiation

Definition (Derivative at the point $x = a$)

$$\frac{d}{dx}f(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This can be explained graphically:



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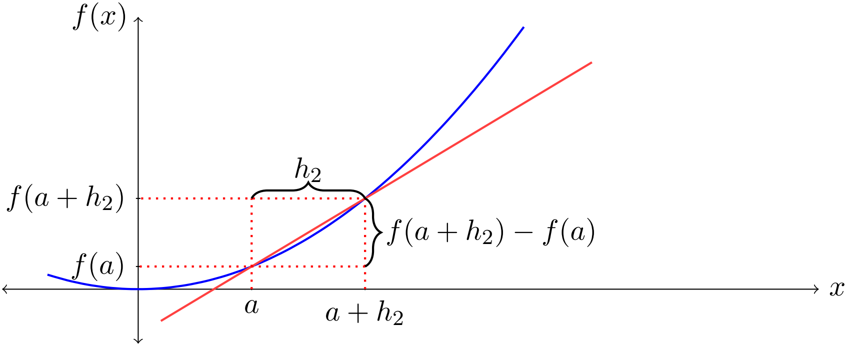

Formal Statement of Differentiation

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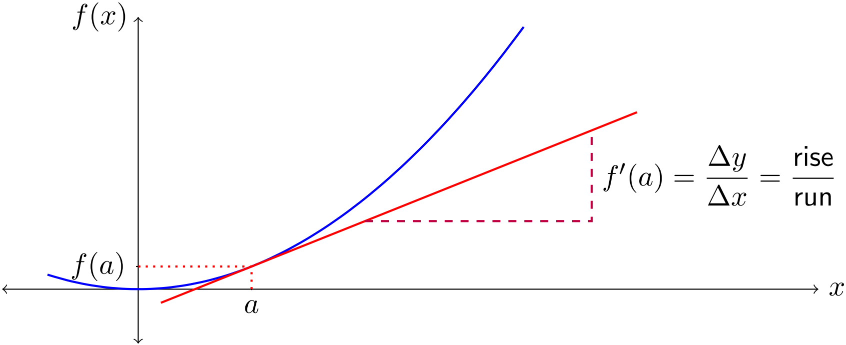
Definition (Derivative at the point $x = a$)

$$\frac{d}{dx}f(a) = f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \approx \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

This can be explained graphically:



Example using the technical definition

Example (Differentiate $f(x) = x^2$ by first principles)

Using the definition on the previous slide:

$$\begin{aligned}
\frac{d}{dx}f(x) = f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} (2x+h) \\
&= 2x
\end{aligned}$$

Notation

- Given some function of x, $y = f(x)$, the following expressions are equivalent:

$$\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x) = f'(x).$$

- We read $\frac{dy}{dx}$ as "the derivative of y with respect to x."
- We can differentiate with respect to whatever variable we'd like. For example if y is a function of u, $y = f(u)$, we can differentiate y with respect to u:

$$\frac{dy}{du} = f'(u).$$

- We read $f'(x)$ as f prime x. This notation is often used for convenience when there is no ambiguity about what we are differentiating with respect to.

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Differentiation in practice

Rarely do people differentiate by "first principles", i.e. using the definition. Instead, we use some simple rules: ▸ More

Function: $f(x) = y$	Derivative $f'(x) = \frac{dy}{dx}$
x^n	nx^{n-1}
ax^n	anx^{n-1}
a (some constant)	0
$\log(x)$	$x^{-1} = \frac{1}{x}$
e^x	e^x

Example ($f(x) = x^2$)

Using the above rules,

$$f'(x) = \frac{d}{dx}f(x) = 2x^{2-1} = 2x^1 = 2x.$$

More complicated example

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Example ($f(x) = 2x^4 + 5x^3$)

$$\begin{aligned} f'(x) &= 2 \times 4x^{4-1} + 5 \times 3x^{3-1} \\ &= 8x^3 + 15x^2 \end{aligned}$$

Example (Your turn: $f(x) = 9x^2 + x^3$)

$$\begin{aligned} f'(x) &= \\ &= \\ &= \end{aligned}$$

More complicated example

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Example (Your turn: $f(x) = 9x^2 + x^3$)

$$\begin{aligned} f'(x) &= 9 \times 2x^{2-1} + 3x^{3-1} \\ &= 18x + 3x^2 \\ &= 3x(6 + x) \end{aligned}$$

Differentiating Exponential Functions

- The exponential function is unique in that it is equal to its derivative:

$$\frac{d}{dx}e^{x} = e^{x}$$

- The exponential function is sometimes written as:

$$e^{f(x)} = \exp\{f(x)\}.$$

- In this definition, a function of x, $f(x)$, is exponentiated.
- The rule for finding a derivative of this type is:

$$\frac{d}{dx}\exp\{f(x)\} = f'(x)\exp\{f(x)\}.$$

Differentiating Exponential Functions

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Example ($g(x) = \exp\{2x\}$)

1. Rewrite as $g(x) = \exp\{f(x)\}$ where $f(x) = 2x$.
2. Find, $f'(x) = 2x^{1-1} = 2x^0 = 2$.
3. Thus, $g'(x) = f'(x)\exp\{f(x)\} = 2\exp\{2x\}$.

Differentiating Exponential Functions

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Example (Your turn: $h(x) = \exp\{3x + 1\}$)

1. Rewrite as $h(x) = \exp\{f(x)\}$ where $f(x) =$.
2. Find $f'(x) =$.
3. Thus, $h'(x) = f'(x)\exp\{f(x)\} =$.

Differentiating Exponential Functions

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Differentiating Exponential Functions

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Example (Your turn: $h(x) = \exp\{3x + 1\}$)

1. Rewrite as $h(x) = \exp\{f(x)\}$ where $f(x) = 3x + 1$.
2. Find $f'(x) = 3x^{1-1} + 0 = 3$.
3. Thus, $h'(x) = f'(x)\exp\{f(x)\} =$.

Differentiating Exponential Functions

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Example ($g(x) = \exp\{2x\}$)

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1. Rewrite as $h(x) = \exp\{f(x)\}$ where $f(x) = 3x + 1$.
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3. Thus, $h'(x) = f'(x)\exp\{f(x)\} = 3\exp\{3x + 1\}$.

Differentiating Logarithmic Functions

- As with the exponential function, there are some special rules for differentiating logarithmic (or log) functions.
- Simple case:
$$\frac{d}{dx}\log(x) = \frac{1}{x}$$
- General case:
$$\frac{d}{dx}\log(f(x)) = \frac{f'(x)}{f(x)}$$

Example $(g(x) = \log(2x^2 + x))$

1. Rewrite as $g(x) = \log(f(x))$ where $f(x) = 2x^2 + x$.
2. Find $f'(x) = 4x + 1$.
3. Thus, $g'(x) = \dfrac{f'(x)}{f(x)} = \dfrac{4x+1}{2x^2+x}$.

Differentiating Logarithmic Functions

Example (Your turn: $y = \log(10x^2 - 3x^2)$)

1. Rewrite as $y = \log\{f(x)\}$ where $f(x) =$.
2. Find $f'(x) =$.
3. Thus,

$$\frac{dy}{dx} = \frac{d}{dx}\log(10x^2 - 3x^2) = \frac{f'(x)}{f(x)}$$

$$=$$

$$=$$

$$= \quad .$$

Differentiating Logarithmic Functions

Example (Your turn: $y = \log(10x^2 - 3x^2)$)

1. Rewrite as $y = \log\{f(x)\}$ where $f(x) = 10x^2 - 3x^3$.
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$$=$$

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Differentiating Logarithmic Functions

Example (Your turn: $y = \log(10x^2 - 3x^2)$)

1. Rewrite as $y = \log\{f(x)\}$ where $f(x) = 10x^2 - 3x^3$.
2. Find $f'(x) = 10 \times 2x - 3 \times 3x^2 = 20x - 9x^2 = x(20 - 9x)$.
3. Thus,

$$\frac{dy}{dx} = \frac{d}{dx}\log(10x^2 - 3x^2) = \frac{f'(x)}{f(x)}$$

$$=$$

$$=$$

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Differentiating Logarithmic Functions

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Example (Your turn: $y = \log(10x^2 - 3x^2)$)

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2. Find $f'(x) = 10 \times 2x - 3 \times 3x^2 = 20x - 9x^2 = x(20 - 9x)$.
3. Thus,

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}\log(10x^2 - 3x^2) &= \frac{f'(x)}{f(x)} \\ &= \frac{x(20-9x)}{10x^2 - 3x^3} \\ &= \frac{x(20-9x)}{x^2(10-3x)} \\ &= \frac{20-9x}{x(10-3x)}.\end{aligned}$$

Other useful differentiation rules

Definition (Specialised Chain Rule)

Let $y = [f(x)]^n$,

$$\frac{dy}{dx} = nf'(x)\,[f(x)]^{n-1}$$

- This is a special case of the chain rule

More

Example ($y = (x^2 + 2)^3$)

Here $y = [f(x)]^n = (x^2 + 2)^3$ so

- $f(x) = x^2 + 2 \implies f'(x) = \frac{d}{dx}(x^2 + 2) = 2x$
- $n = 3$
- $\frac{dy}{dx} = nf'(x)\,[f(x)]^{n-1} = 3 \times 2x \times (x^2 + 2)^{3-1} = 6x(x^2 + 2)^2$

Other useful differentiation rules

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Definition (Product Rule)

Let $y = uv$ where u and v are functions of x,

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} = uv' + vu'$$

▸ More

Example ($y = x^3 \log(x)$)

Here $u = x^3$; $v = \log(x)$; $v' = \dfrac{dv}{dx} = \dfrac{1}{x}$; $u' = \dfrac{du}{dx} = 3x^2$ so

$$\begin{aligned}\frac{dy}{dx} &= u \times v' + v \times u' \\ &= x^3 \times \frac{1}{x} + \log(x) \times 3x^2 \\ &= x^2 + 3x^2 \log(x) \\ &= x^2(1 + 3\log(x)).\end{aligned}$$

Your turn...

Example (Your turn $y = (x^3 - 1)^2$)

Think of our function as $y = [f(x)]^n$ we have in this particular case,

- $f(x) =$
- $f'(x) =$
- $n =$

So,

$$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$$
$$=$$
$$=$$

Your turn. . .

Example (Your turn $y = (x^3 - 1)^2$)

Think of our function as $y = [f(x)]^n$ we have in this particular case,

- $f(x) = x^3 - 1$
- $f'(x) =$
- $n =$

So,

$$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$$
$$=$$
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Your turn...

Example (Your turn $y = (x^3 - 1)^2$)

Think of our function as $y = [f(x)]^n$ we have in this particular case,

- $f(x) = x^3 - 1$
- $f'(x) = \dfrac{d}{dx}(x^3 - 1) = 3x^2$
- $n =$

So,

$$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$$
$$=$$
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Your turn...

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- $f(x) = x^3 - 1$
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- $n = 2$

So,

$$\begin{aligned} \frac{dy}{dx} &= nf'(x)\left[f(x)\right]^{n-1} \\ &= \\ &= \end{aligned}$$

Your turn...

Example (Your turn $y = (x^3 - 1)^2$)

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- $f(x) = x^3 - 1$
- $f'(x) = \frac{d}{dx}(x^3 - 1) = 3x^2$
- $n = 2$

So,

$$\begin{aligned} \frac{dy}{dx} &= nf'(x)\,[f(x)]^{n-1} \\ &= 2 \times 3x^2 \times (x^3 - 1)^{2-1} \\ &= 6x^2(x^3 - 1) \end{aligned}$$

Your turn. . .

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Example (Your turn $y = 2x^{-2}\log(x-1)$)

We're differentiating a product so think of the function as $y = uv$.

- Let $u =$ which means $u' = \dfrac{du}{dx} =$
- Let $v =$ which means $v' = \dfrac{dv}{dx} =$

So,

$$\frac{dy}{dx} = uv' + vu' =$$

$$=$$

Note that you can always check you differentiation using WolframAlpha: `d/dx 2x^(-2)*log(x-1)`

More

Your turn. . .

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More

Your turn...

Example (Your turn $y = 2x^{-2}\log(x-1)$)

We're differentiating a product so think of the function as $y = uv$.

- Let $u = 2x^{-2}$ which means $u' = \dfrac{du}{dx} = -4x^{-3}$
- Let $v =$ which means $v' = \dfrac{dv}{dx} =$

So,

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$$=$$

Note that you can always check you differentiation using WolframAlpha: `d/dx 2x^(-2)*log(x-1)`

More

Your turn. . .

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So,

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More

Your turn...

Example (Your turn $y = 2x^{-2}\log(x-1)$)

We're differentiating a product so think of the function as $y = uv$.

- Let $u = 2x^{-2}$ which means $u' = \dfrac{du}{dx} = -4x^{-3}$
- Let $v = \log(x-1)$ which means $v' = \dfrac{dv}{dx} = \dfrac{1}{x-1}$

So,

$$\frac{dy}{dx} = uv' + vu' =$$

$$=$$

Note that you can always check you differentiation using WolframAlpha: `d/dx 2x^(-2)*log(x-1)`

More

Your turn. . .

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We're differentiating a product so think of the function as $y = uv$.

- Let $u = 2x^{-2}$ which means $u' = \dfrac{du}{dx} = \; -4x^{-3}$
- Let $v = \log(x-1)$ which means $v' = \dfrac{dv}{dx} = \dfrac{1}{x-1}$

So,

$$\frac{dy}{dx} = uv' + vu' = 2x^{-2}\frac{1}{x-1} + (-4x^{-3})\log(x-1)$$

$$=$$

Note that you can always check you differentiation using WolframAlpha: `d/dx 2x^(-2)*log(x-1)`

More

Your turn. . .

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Example (Your turn $y = 2x^{-2}\log(x-1)$)

We're differentiating a product so think of the function as $y = uv$.

- Let $u = 2x^{-2}$ which means $u' = \dfrac{du}{dx} = -4x^{-3}$
- Let $v = \log(x-1)$ which means $v' = \dfrac{dv}{dx} = \dfrac{1}{x-1}$

So,

$$\begin{aligned}\frac{dy}{dx} = uv' + vu' &= 2x^{-2}\frac{1}{x-1} + (-4x^{-3})\log(x-1)\\ &= \frac{2}{x^2(x-1)} - 4x^{-3}\log(x-1)\end{aligned}$$

Note that you can always check you differentiation using WolframAlpha: `d/dx 2x^(-2)*log(x-1)`

▸ More

Other useful differentiation rules

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Definition (Quotient Rule)

Let $y = \dfrac{u}{v}$ where u and v are functions of x then

$$\frac{dy}{dx} = \frac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

▸ More

Example ($y = \dfrac{x^4}{e^x}$)

Let $u = x^4$ and $v = e^x$; then $u' = \dfrac{du}{dx} = 4x^3$ and $v' = \dfrac{dv}{dx} = e^x$,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{e^x \cdot 4x^3 - x^4 \cdot e^x}{(e^x)^2} = \frac{x^3(4-1)}{e^x}.$$

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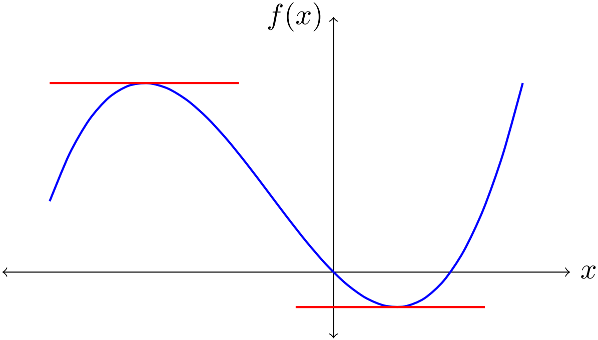
Finding Maxima and Minima

The most common use of differentiation is to find the maximum and minimum values of functions.

Key Idea

"Stationary points" occur when the derivative equals zero, $f'(x) = 0$, i.e. the tangent line is a horizontal line. ▸ More



Finding Maxima and Minima

To determine if a stationary point is a maximum, minimum or neither, we find the second order derivatives.

Definition (Second order derivative)

The second order derivative of a function, $f(x)$, is found by taking the derivative of the first order derivative:

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right).$$

- If $f''(x) < 0$, the stationary point at x is a maximum.
- If $f''(x) > 0$, the stationary point at x is a minimum.
- If $f''(x) = 0$, the nature of the stationary point must be determined by way of other means, often by noting a sign change around that point.

Finding Turning Points

Example ($f(x) = 2x^4 + 5x^3$)

We found previously that $f'(x) = 8x^3 + 15x^2 = x^2(8x + 15)$.

- To find the turning points, we set $f'(x) = 0$:

$$x^2(8x + 15) = 0$$

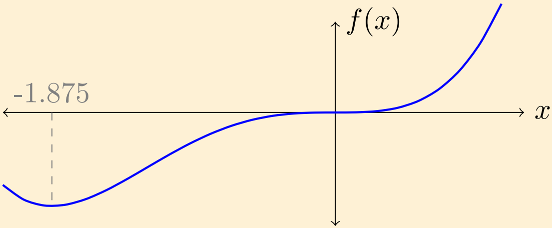
- This occurs when either:
  - $x^2 = 0 \implies x = 0$ (this is a point of inflection) ▸ More
  - $8x + 15 = 0 \implies x = -\dfrac{15}{8} = -1.875$ (a minimum) ▸ More
- To determine whether these are turning points are maxima, minima or neither we find the second order derivative:

$$\begin{aligned} f''(x) = \frac{d}{dx} f'(x) &= \frac{d}{dx}\left(8x^3 + 15x^2\right) \\ &= 24x^2 + 30x \end{aligned}$$

Finding Turning Points

Example ($f(x) = 2x^4 + 5x^3$ continued)

- The second order derivative is: $f''(x) = 24x^2 + 30x$
- We need to evaluate $f''(x)$ at the values of x we identified as turning points:
  - $f''(0) = 0 \implies$ a point of inflection ▸ More
  - $f''(-1.875) = 24 \times (-1.875)^2 - 30 \times 1.875 = 28.125 > 0$ $\implies$ a minimum ▸ More



Your turn to find Maxima and Minima. . .

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Given $f(x) = 9x^2 + x^3$, previously you found $f'(x) = 18x + 3x^2$.

- To find the turning points set $f'(x) = 0$:
- This occurs when either:
  - or
  -
- Second order derivative: $f''(x) =$. Evaluate this at the possible turning points:
  -
  -

Your turn to find Maxima and Minima. . .

Given $f(x) = 9x^2 + x^3$, previously you found $f'(x) = 18x + 3x^2$.

- To find the turning points set $f'(x) = 0$:

$$3x(6 + x) = 0$$

- This occurs when either:
  - $3x = 0 \implies x = 0$ or
  - $6 + x = 0 \implies x = -6$
- Second order derivative: $f''(x) =$. Evaluate this at the possible turning points:
  -
  -

Your turn to find Maxima and Minima...

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- Second order derivative: $f''(x) = 18 + 6x$. Evaluate this at the possible turning points:
  -
  -

Your turn to find Maxima and Minima...

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$$3x(6 + x) = 0$$

- This occurs when either:
  - $3x = 0 \implies x = 0$ or
  - $6 + x = 0 \implies x = -6$
- Second order derivative: $f''(x) = 18 + 6x$. Evaluate this at the possible turning points:
  - $f''(0) = 18 > 0 \implies$
  - $f''(-6) = 18 + 6 \times (-6) = -18 < 0 \implies$

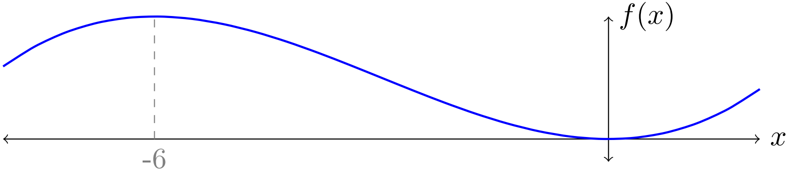
Your turn to find Maxima and Minima...

Given $f(x) = 9x^2 + x^3$, previously you found $f'(x) = 18x + 3x^2$.

- To find the turning points set $f'(x) = 0$:

$$3x(6 + x) = 0$$

- This occurs when either:
  - $3x = 0 \implies x = 0$ or
  - $6 + x = 0 \implies x = -6$
- Second order derivative: $f''(x) = 18 + 6x$. Evaluate this at the possible turning points:
  - $f''(0) = 18 > 0 \implies$ a minimum
  - $f''(-6) = 18 + 6 \times (-6) = -18 < 0 \implies$ a maximum



Maximising Utility

- An investor gains what is known as utility from increasing his/her wealth (think of utility as simply, enjoyment).
- You can define someones utility as a function of wealth.

Example (Your turn: $U(w) = 4w - \frac{1}{10}w^2$)

- Differentiate U with respect to w:
- Set $U'(w) = 0$ to find the critical points:
- $w =$ gives the theoretical level of wealth for this investor that will maximise their utility.

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Applications in Business

- The ubiquitous Cobb-Douglas production function uses exponentials and logs ▸ More
- The formal interpretation of regression coefficients in econometrics requires differentiation ▸ More
- Differentiation to finding maxima is used for constrained optimisation in operations management ▸ More
- Marginal benefits and marginal costs can be derived using differentiation ▸ More

Summary

- Functions, log and exponential functions
- Differentiation tells us about the behaviour of the function
- The derivative of a single variable function is the tangent
- The derivative can be interpreted as the "rate of change" of the function
- Chain rule, product rule, quotient rule
- Finding maxima and minima
- Applications in Business

Summary of Differentiation Identities

Function	Derivative
$f(x) = ax^n$	$f'(x) = anx^{n-1}$
$f(x) = a$ (some constant)	$f'(x) = 0$
$f(x) = \exp\{g(x)\}$	$f'(x) = g'(x) \cdot \exp\{g(x)\}$
$y = \log\{f(x)\}$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = f(u)$, $u = g(x)$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$y = uv$, $u = g(x)$, $v = h(x)$	$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$
$y = \frac{u}{v}$, $u = g(x)$, $v = h(x)$	$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

▸ More

Additional Resources

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- Test your knowledge at the University of Sydney Business School MathQuiz:
  `http://quiz.econ.usyd.edu.au/mathquiz`
- Additional resources on the Maths in Business website
  `sydney.edu.au/business/learning/students/maths`
- The University of Sydney Mathematics Learning Centre has a number of additional resources:
  - Maths Learning Centre algebra workshop notes ▸ More
  - Other Maths Learning Centre Resources ▸ More
- The Department of Mathematical Sciences and the Mathematics Learning Support Centre at Loughborough University have a fantastic website full of resources. ▸ More
- There's also tonnes of theory, worked questions and additional practice questions online. All you need to do is Google the topic you need more practice with! ▸ More

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- Questions, comments, feedback? Let us know at `business.maths@sydney.edu.au`