MATHS WORKSHOPS

Simultaneous Equations and Inequalities

Business School

THE UNIVERSITY OF SYDNEY

Outline

Recap of Algebra, Linear and Quadratic Functions

Simultaneous Equations

Inequalities

Applications in Business

Summary and Conclusion

Outline

Recap of Algebra, Linear and Quadratic Functions

Simultaneous Equations

Inequalities

Applications in Business

Summary and Conclusion

Variables, Parameters & Solving Equations

THE UNIVERSITY OF SYDNEY

Definition (Parameters)

A parameter is some fixed value, also known as a "constant" or "coefficient."

▸ More

Definition (Variables)

A variable is an unknown value that may change, or vary, depending on the parameter values.

▸ More

Definition (Solving an equation)

We can solve an equation by using mathematical operations to rearrange the equation such that the variable is on one side of the equation and the parameters are all on the other side. Example: $x = \dfrac{c-b}{a}$.

▸ More

Linear functions

THE UNIVERSITY OF SYDNEY

Definition (Linear function)

An equation with two variables of the form $y = ax + b$ is called a linear function.

More

Definition (Independent and dependent variables)

The variable on the right hand side of the equation, x, is called the independent variable and the variable on the left hand side of the equation, y, is called the dependent variable.

- The dependent variable may also be written as $y = f(x)$ or $y = g(x)$.
- This notation emphasises that y is a function of x, in other words y depends on x.

More

Outline

Recap of Algebra, Linear and Quadratic Functions

Simultaneous Equations

Inequalities

Applications in Business

Summary and Conclusion

Simultaneous Equations

THE UNIVERSITY OF SYDNEY

Definition (Simultaneous Equations)

If two equations are both "true" at the same time, they are called simultaneous equations.

More

Example

A system of two simultaneous equations:

$$y = 4x$$
$$2x + y = 6$$

Definition (Solution)

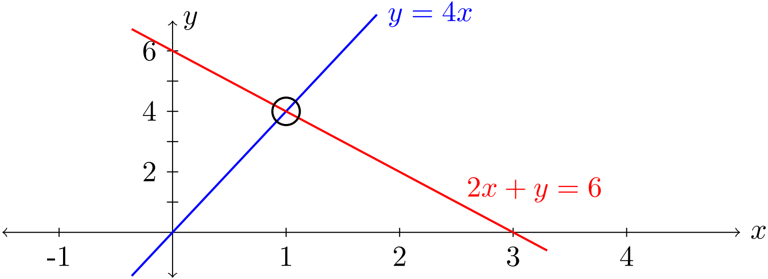
To solve a system of simultaneous equations we need to find values of the variables that satisfy all equations in the system.

Simultaneous Equations

Definition (Solution)

To solve a system of simultaneous equations we need to find values of the variables that satisfy all equations in the system.

Graphically this is the point where the two lines cross:



How to solve systems of equations?

The general approach consists of 3 steps:

1. Manipulate the equations to find an expression in terms of one variable only.
2. Solve the equation for that one variable
3. Use that solution in one of the original equations to find the other solution.

There are two main ways to manipluate the equations in step 1:

Definition (Substitution Method)

Substitute one equation into another. More

Definition (Elimination Method)

Add or subtract a multiple of one equation from the other. More

Substitution Method

We can use the 3 step approach to solve the following system:

$$y = 4x \quad (1)$$

$$2x + y = 6 \quad (2)$$

1. Substitute Equation (1) into Equation (2):

$$2x + 4x = 6 \qquad \text{(substituting } y = 4x\text{)}$$

$$6x = 6$$

2. Solve this equation for x:

$$6x \times \frac{1}{6} = 6 \times \frac{1}{6} \qquad \text{(divide both sides by 6)}$$

$$x = 1$$

3. Use this solution, $x = 1$, in Equation (1) to find y:

$$y = 4x = 4 \times 1 = 4$$

Elimination Method

3 step approach using the elimination method:

$$y = 4x \tag{1}$$
$$2x + y = 6 \tag{2}$$

1. Eliminate y in Equation (2) by subtracting (1) from (2):

$$2x + y - y = 6 - 4x$$
$$2x = 6 - 4x \qquad \text{(no longer any } y\text{'s)}$$
$$6x = 6$$

2. Using exactly the same approach as in the substitution method we solve to find $x = 1$.
3. As before, we substitute $x = 1$ back into Equation (1) to find $y = 4$.

Simultaneous Equations Your Turn. . .

Solve the following system of equations

$$2x + y = 8 \qquad (3)$$

$$x + y = 6 \qquad (4)$$

1.

2.

3.

Simultaneous Equations Your Turn...

Solve the following system of equations

$$2x + y = 8 \quad (3)$$

$$x + y = 6 \quad (4)$$

1. The elimination method and subtract (4) from (3):

   OR the substitution method by rearranging (4) to get $y = 6 - x$ and substituting this into (3):

2.
3.

Simultaneous Equations Your Turn. . .

Solve the following system of equations

$$2x + y = 8 \tag{3}$$
$$x + y = 6 \tag{4}$$

1. The elimination method and subtract (4) from (3):

$$2x + y - (x + y) = 8 - 6$$
$$x = 2$$

OR the substitution method by rearranging (4) to get $y = 6 - x$ and substituting this into (3):

$$2x + (6 - x) = 8$$
$$x = 2$$

2.
3.

Simultaneous Equations Your Turn. . .

Solve the following system of equations

$$2x + y = 8 \quad (3)$$

$$x + y = 6 \quad (4)$$

1. The elimination method and subtract (4) from (3):

$$2x + y - (x + y) = 8 - 6$$

$$x = 2$$

   OR the substitution method by rearranging (4) to get $y = 6 - x$ and substituting this into (3):

$$2x + (6 - x) = 8$$

$$x = 2$$

2. No need to solve for x as we can see directly that $x = 2$.
3. Use $x = 2$ in (4) to find $y = 6 - 2 = 4$.

Graphical Example

- In the Algebra Workshop we showed how to graph linear functions.
- The solution of a system of equations can be graphically represented as the point of intersection of the two equations.

Definition (Intersection)

The intersection is the point at which two lines cross.

Definition (Cartesian coordinates)

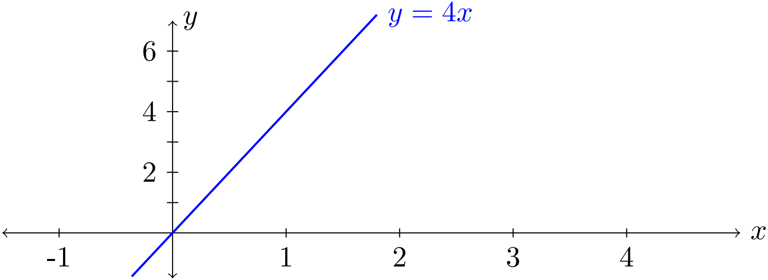
It is sometimes useful to use the cartesian coordinate system to refer to points in the 2-dimensional plane. Instead of writing $x = 2$ and $y = 4$ we instead write as $(x, y) = (2, 4)$ or just refer to the point $(2, 4)$.

More

Graphing simultaneous equations

$$y = 4x$$
$$2x + y = 6$$

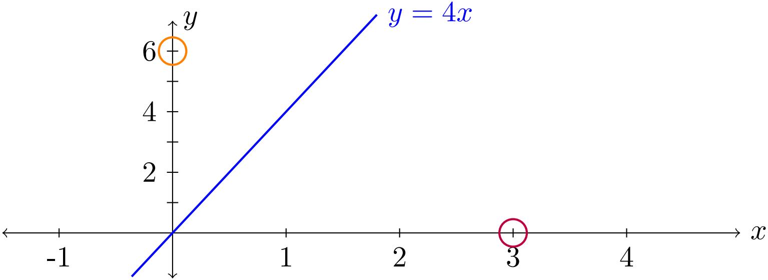
- $y = 4x$ is simple to plot, it goes through the origin $(x, y) = (0, 0)$ and has slope equal to 4.



Graphing simultaneous equations

$$y = 4x$$
$$2x + y = 6$$

- $y = 4x$ is simple to plot, it goes through the origin $(x, y) = (0, 0)$ and has slope equal to 4.
- $2x + y = 6$ is a bit tricky. When $x = 0$ the intercept is $y = 6$ and when $y = 0 \implies 2x = 6$ or $x = 3$, so the line passes through the two points (0,6) and (3,0):

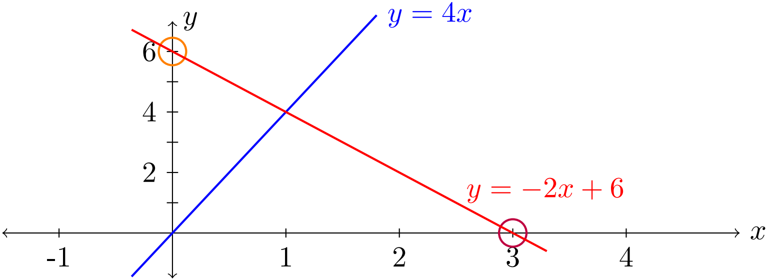


Graphing simultaneous equations

THE UNIVERSITY OF SYDNEY

$$y = 4x$$
$$2x + y = 6$$

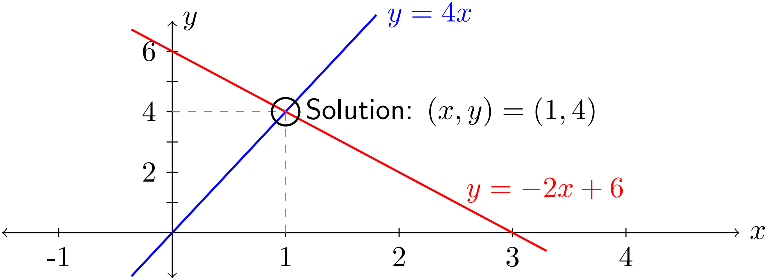
- $y = 4x$ is simple to plot, it goes through the origin $(x, y) = (0, 0)$ and has slope equal to 4.
- $2x + y = 6$ is a bit tricky. When $x = 0$ the intercept is $y = 6$ and when $y = 0 \implies 2x = 6$ or $x = 3$, so the line passes through the two points (0,6) and (3,0):



Graphing simultaneous equations

$$y = 4x$$
$$2x + y = 6$$

- $y = 4x$ is simple to plot, it goes through the origin $(x, y) = (0, 0)$ and has slope equal to 4.
- $2x + y = 6$ is a bit tricky. When $x = 0$ the intercept is $y = 6$ and when $y = 0 \implies 2x = 6$ or $x = 3$, so the line passes through the two points (0,6) and (3,0):

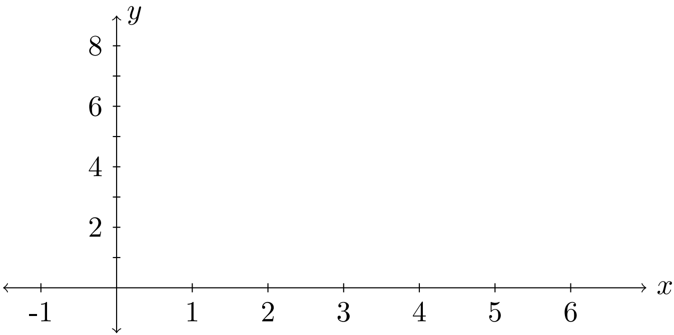


Now it's your turn...

Find the solution to this system of equations graphically

$$2x + y = 8 \quad (3)$$

$$x + y = 6 \quad (4)$$



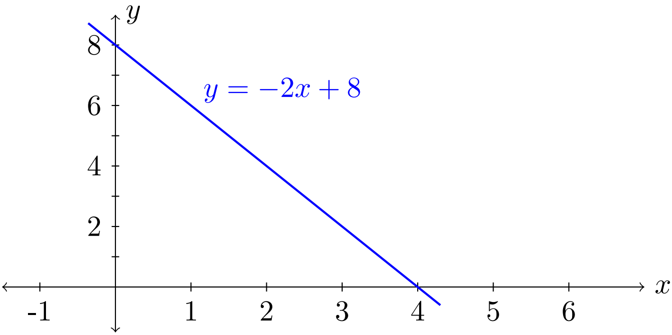
Now it's your turn. . .

Find the solution to this system of equations graphically

$$2x + y = 8 \quad (3)$$

$$x + y = 6 \quad (4)$$

1. Consider Equation (3). When $x = 0$, the intercept is $y = 8$. When $y = 0$, $2x = 8 \implies x = 4$.



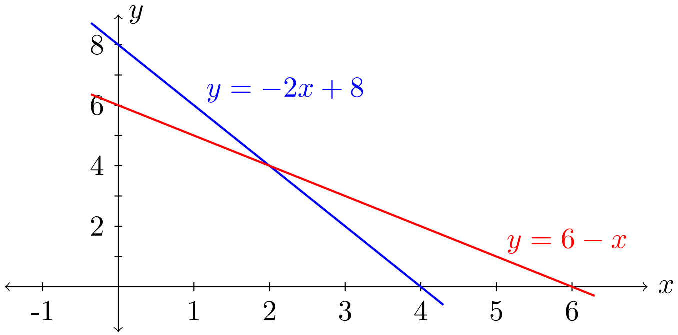
Now it's your turn...

Find the solution to this system of equations graphically

$$2x + y = 8 \tag{3}$$

$$x + y = 6 \tag{4}$$

2. Consider Equation (4). When $x = 0$, the intercept is $y = 6$. When $y = 0$, $x = 6$.



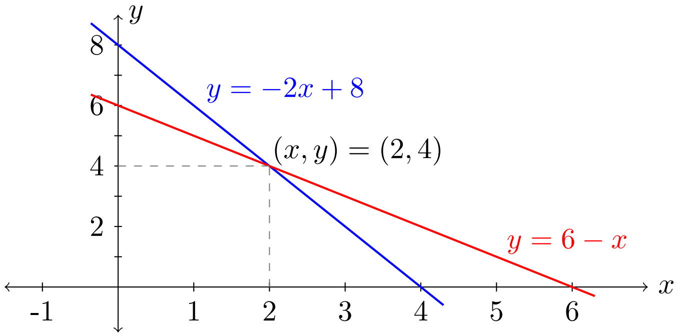
Now it's your turn...

Find the solution to this system of equations graphically

$$2x + y = 8 \qquad (3)$$

$$x + y = 6 \qquad (4)$$

3. The solution is where the lines intersect. In this case, $x = 2$ and $y = 4$ just like we found algebraically.



How many solutions?

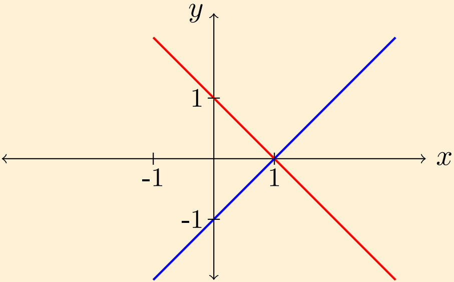
A system of linear equations can have:

- Exactly one solution (intersecting lines)
- No solutions (parallel lines)
- Infinitely many solutions (same line)

Example (Exactly one solution)

$y = 1 - x$

$y = -1 + x$



More

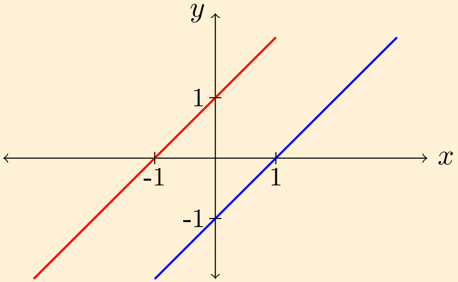
How many solutions?

A system of linear equations can have:

- Exactly one solution (intersecting lines)
- No solutions (parallel lines)
- Infinitely many solutions (same line)

Example (No Solutions)

$$y = x + 1$$
$$y = x - 1$$



More

How many solutions?

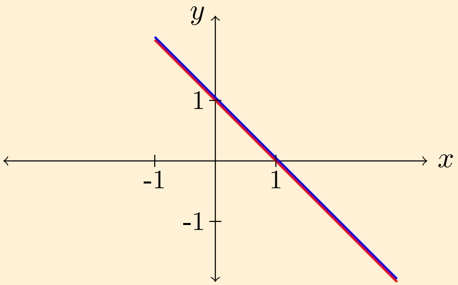
A system of linear equations can have:

- Exactly one solution (intersecting lines)
- No solutions (parallel lines)
- Infinitely many solutions (same line)

Example (Infinitely many solutions)

$y = 1 - x$

$2y = 2 - 2x$



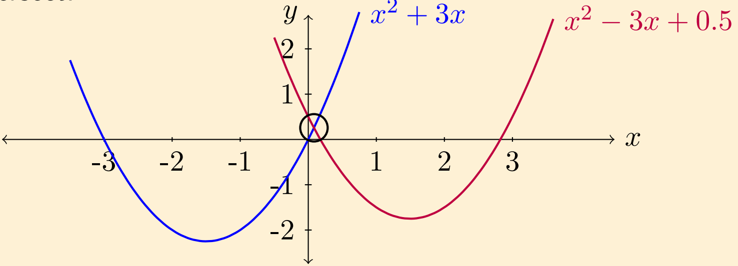
Simultaneous equations with quadratics

Example (Simultaneous Quadratic Equations)

$$y = x^2 + 3x \quad (1)$$

$$y = x^2 - 3x + 0.5 \quad (2)$$

Graphically the solution is the point where the two functions intersect:



Simultaneous equations with quadratics

Example (Simultaneous Quadratic Equations)

$$y = x^2 + 3x \quad (1)$$
$$y = x^2 - 3x + 0.5 \quad (2)$$

Algebraically, we can set the two equations equal to one another, (1) = (2), and solve for x:

$$\begin{aligned} x^2 + 3x &= x^2 - 3x + 0.5 \\ x^2 - x^2 + 3x + 3x &= 0.5 \\ 6x &= 0.5 \\ x &= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}. \end{aligned}$$

From (1), $y = x^2 + 3x = \left(\frac{1}{12}\right)^2 + 3 \times \frac{1}{12} = \frac{37}{144} = 0.2569.$

Outline

Recap of Algebra, Linear and Quadratic Functions

Simultaneous Equations

Inequalities

Applications in Business

Summary and Conclusion

Inequalities

THE UNIVERSITY OF SYDNEY

Definition (Inequality)

In mathematics, an inequality is a statement about the relative size of two objects, or about whether they are the same or not.

Example (Strict Inequalities)

- $a < b$ means that a is less than b
- $a > b$ means that a is greater than b
- $a \neq b$ means that a is not equal to b

Example (Not Strict Inequalities)

- $a \leq b$ means that a is less than or equal to b
- $a \geq b$ means that a is greater than or equal to b

More

Inequalities

Example

$$2 - \frac{7x}{5} > -x + 3$$

$$10 - 7x > -5x + 15 \qquad \text{(multiply both sides by 5)}$$

$$-7x > -5x + 15 - 10 \qquad \text{(subtract 10 from both sides)}$$

$$-7x + 5x > 5 \qquad \text{(add } 5x \text{ to both sides)}$$

$$-2x > 5$$

$$x < -\frac{5}{2} \qquad \text{(divide both sides by } -2\text{)}$$

- Look what happened in the last step we divided through by a negative number – we flipped the inequality!!!

Flipping the inequality

THE UNIVERSITY OF SYDNEY

The hardest thing about inequalities is remembering to flip the inequality when you multiply through or divide through by -1.

- Consider: $-2 < 5$. That statement is true (right?)
- If we multiply both sides by -1 we would get $2 < -5$.
  - This is is clearly wrong (right?) $2 \not< -5$
- In fact, $2 > -5$.

Rule

If you multiply or divide an inequality by a negative number you must reverse the sign of the inequality!

More

Rule

If a and b are both positive or both negative and you take the reciprocal of both sides:

$$a > b \implies \frac{1}{a} < \frac{1}{b}$$

But WHY?

Consider some $a > b$:

$$\begin{aligned} a > b &\implies a - b > 0 && \text{(subtracting } b \text{ from both sides)} \\ &\implies -b > -a && \text{(subtracting } a \text{ from both sides)} \\ &\implies -a < -b && \text{(rewriting the inequality in reverse)} \end{aligned}$$

Again consider some $a > b$, where both a and b are positive numbers:

$$\begin{aligned} a > b &\implies 1 > \frac{b}{a} && \text{(dividing both sides by } a\text{)} \\ &\implies \frac{1}{b} > \frac{1}{a} && \text{(dividing both sides by } b\text{)} \\ &\implies \frac{1}{a} < \frac{1}{b} && \text{(rewriting the inequality in reverse)} \end{aligned}$$

Using Inequalities

Example (Your Turn)

One of the colleges on campus is organising informal with all drinks included in the ticket price. The DJ costs \$300 for the night and the bouncers charge \$500 for the night. The drink expenses for each guest is \$20. How many people need to attend before the college starts making money if the tickets are \$30 per person?

- In order to make money we need our income to be bigger than the expenses:

$$\text{Income} > \text{Expenses}$$

Using Inequalities

Example (Your Turn)

One of the colleges on campus is organising informal with all drinks included in the ticket price. The DJ costs \$300 for the night and the bouncers charge \$500 for the night. The drink expenses for each guest is \$20. How many people need to attend before the college starts making money if the tickets are \$30 per person?

- In order to make money we need our income to be bigger than the expenses:

$$\begin{aligned} \text{Income} &> \text{Expenses} \\ 30x &> 20x + 300 + 500 \\ 10x &> 800 \\ x &> 80 \end{aligned}$$

- So we need more than 80 people to attend!

Quadratic Inequalities

Example (Your Turn)

Solve $x^2 - 3x - 4 < 0$ and graph the solution set on a number line.

1. Factorise:

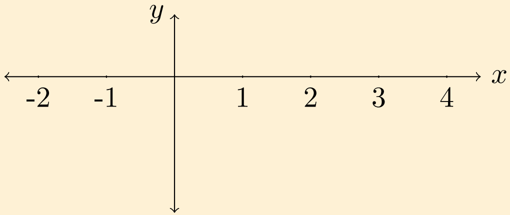
Quadratic Inequalities

THE UNIVERSITY OF SYDNEY

Example (Your Turn)

Solve $x^2 - 3x - 4 < 0$ and graph the solution set on a number line.

1. Factorise: $(x-4)(x+1) < 0$
2. Graph:



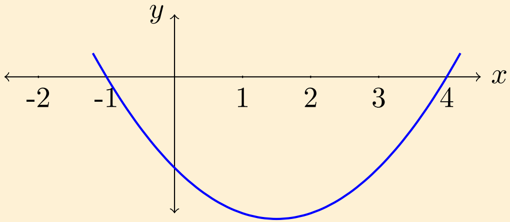
Quadratic Inequalities

THE UNIVERSITY OF SYDNEY

Example (Your Turn)

Solve $x^2 - 3x - 4 < 0$ and graph the solution set on a number line.

1. Factorise: $(x-4)(x+1) < 0$
2. Graph:



3. Determine where the function is negative and where it is positive

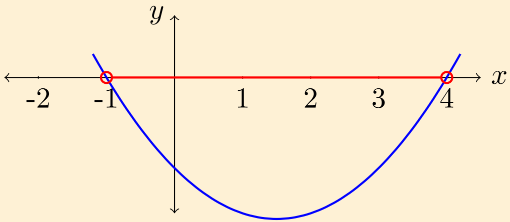
Quadratic Inequalities

THE UNIVERSITY OF SYDNEY

Example (Your Turn)

Solve $x^2 - 3x - 4 < 0$ and graph the solution set on a number line.

1. Factorise: $(x-4)(x+1) < 0$
2. Graph:



3. Determine where the function is negative and where it is positive so the solution is $-1 < x < 4$.

Outline

Recap of Algebra, Linear and Quadratic Functions

Simultaneous Equations

Inequalities

Applications in Business

Summary and Conclusion

Equilibrium Demand and Supply

THE UNIVERSITY OF SYDNEY

Equilibrium

The price, P, of a good is related to the quantity, Q, of it demanded and supplied in the market.

- A **demand curve** shows that as price decreases, the quantity demanded of the product increases. Example:

$$P = -2Q + 50$$

- A **supply curve** shows that as price increases, quantity of the product supplied increases. Example:

$$P = 4Q + 5$$

- The point at which **supply** equals **demand** is the **equilibrium** price and quantity.

More

Equilibrium Demand and Supply

Demand curve:

$$P = -2Q + 50 \tag{3}$$

Supply curve:

$$P = 4Q + 5 \tag{4}$$

Example (Your Turn...)

1. Graph the curves and identify the point of intersection (Hint: put price on the y-axis and demand on the x-axis)
2. Find the equilibrium price and quantity (i.e. solve (3) and (4) simultaneously).

Equilibrium Demand and Supply

1. Graph the curves and identify the point of intersection

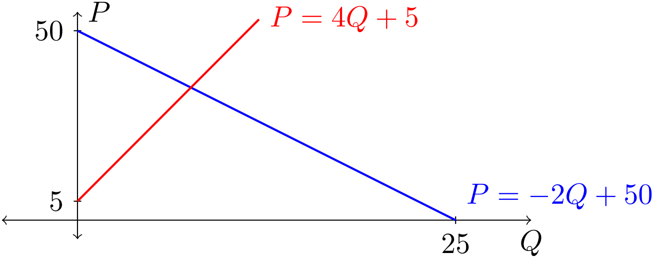
2. To find the equilibrium point, we equate the demand curve and the supply curve. I.e. set (3) = (4):

   To find P we use $Q =$ in either (3) or (4):

   .

Equilibrium Demand and Supply

1. Graph the curves and identify the point of intersection



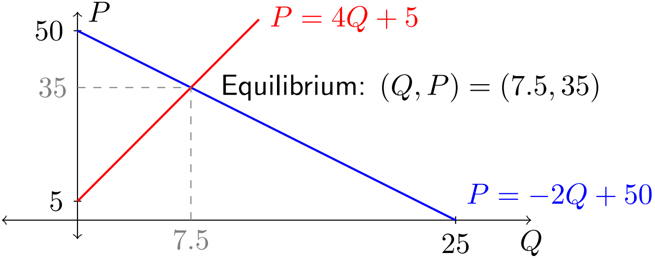
2. To find the equilibrium point, we equate the demand curve and the supply curve. I.e. set (3) = (4):

   To find P we use $Q =$ in either (3) or (4):

   .

Equilibrium Demand and Supply

1. Graph the curves and identify the point of intersection



2. To find the equilibrium point, we equate the demand curve and the supply curve. I.e. set (3) = (4):

$$-2Q + 50 = 4Q + 5 \implies Q = 7.5$$

To find P we use $Q = 7.5$ in either (3) or (4):

$$P = 4Q + 5 = 4 \times 7.5 + 5 = 35.$$

Applications in Business

THE UNIVERSITY OF SYDNEY

Simultaneous Equations

- Systems of simultaneous equations are solved more generally using matrices. ▸ More
- Matrices are fundamental to finding least squares regression estimates in Statistics. ▸ More
- Break-even analysis in Accounting ▸ More

Inequalities

- Constrained optimisation problems in Management Decision Science ▸ More
- Hypothesis testing and constructing confidence intervals in Econometrics ▸ More

Outline

Recap of Algebra, Linear and Quadratic Functions

Simultaneous Equations

Inequalities

Applications in Business

Summary and Conclusion

Summary

- parameters, variables and solving equations
- simultaneous equations
- graphing simultaneous equation
- algebraic solution
- graphical solution
- consistent system of equations vs. inconsistent system
- solving inequalities
- flipping the inequality

Coming up...

Week 6: Differentiation

- Theory and rules of Differentiation
- Differentiating various functions and application of Differentiation

Additional Resources

- Test your knowledge at the University of Sydney Business School MathQuiz:
  `http://quiz.econ.usyd.edu.au/mathquiz`
- Additional resources on the Maths in Business website
  `sydney.edu.au/business/learning/students/maths`
- The University of Sydney Mathematics Learning Centre has a number of additional resources:
  - Basic concepts in probability notes ▸ More
  - Sigma notation notes ▸ More
  - Permutations and combinations notes ▸ More
  - Further workshops by the Maths Learning Centre ▸ More
- There's also tonnes of theory, worked questions and additional practice questions online. All you need to do is Google the topic you need more practice with! ▸ More

Acknowledgements

- Presenters and content contributors: Garth Tarr, Edward Deng, Donna Zhou, Justin Wang, Fayzan Bahktiar, Priyanka Goonetilleke.
- Mathematics Workshops Project Manager Jessica Morr from the Learning and Teaching in Business.
- Valuable comments, feedback and support from Erick Li and Michele Scoufis.

- Questions, comments, feedback? Let us know at `business.maths@sydney.edu.au`